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TOPOLOGICAL INVARIANTS OF VARIETY AND THE NUMBER OF IT'S HOLOMORPHIC MAPPINGS.

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Let X, Y be two complex nonsingular quasiprojective varieties and H(X,Y) the set of all proper surjective holomorphic mappings. This set is finite provided that Y is of hyperbolic type, i.e. the logarithmic Kodaira dimension $\bar{k}(X)$ = dim X ([T],[K-O]). The following question is discussed in this talk.

Question 1. Are there any topological restrictions on the possible number #H(X,Y) of the mappings $f \in H(X,Y)$?

This means that we are looking for the bounds for #H(X,Y) depending on topology of variety X only. In compact case the answer to the Question 1 is affirmative, at least for the varieties X, Y with ample canonical line bundles K(X), K(Y). Parameter $K(X)^{\dim X}$ (or $(-1)^n \ c_1(X)^n$) is the topological invariant in question. Further we shall not make difference between divisors and line bundles corresponding to them if no confusion may arise.

Theorem 1. There is such a function $\varphi: Z \times Q \longrightarrow Z$ that for any pair of nonsingular projective varieties X, Y with ample canonical bundles K(X), K(Y) the number $\#H(X,Y) \leq \varphi\ (\dim X,\ K(X)^{\dim X})$.

This estimate is not effective, but for Riemann surfaces (dim X=1) there is an explicit formula (A. Howard and A.J. Sommese [H-S]) for the upper bound of #H(X.Y) depending on the genus g_X of the surface X only.

Theorem 1 is the immediate consequence of the existing estimate of #H(X,Y), depending on Hilbert polynomial $P_{K(X)} = \chi(X,mK(X))$ of variety X ([B1]) and the Lemma, belonging to J. Kollar and T. Matsusaka ([K-M]).

Lemma (Kollar-Matsusaka). For every n there is a polynomial P(x,y) such that if V^n is a nonsingular projective variety and X a semi-ample divisor with the Hilbert polynomial $\chi(V,\mathcal{O}(rX)) = \sum_0^n d_i r^i$ then $|d_i| \le P(d_n, d_{n-1})$.

In the case considered $d_n = K(X)^n/n!$, $d_{n-1} = K(X)^n/2(n-1)!$. From the Lemma follows that $|P_{K(X)}(m)| \le (n+1)^{n-1}P(d_n, d_{n-1}) = M$ for m=1,...,n+1. Since the Hilbert polynomial is integervalued, there may be at most $s = (2M)^{n+1}$ possible Hilbert polynomials $P_1,\dots,P_s$ with given $d_n = K(X)^n$. For each of P_i there exists a number $b_i(P_i)$, which is the upper bound for #H(X,Y) for all nonsingular projective varieties X, Y with ample canonical line bundles K(X), K(Y) provided $P_{K(X)} = P_i$ ([B1]). We may assume

$$\varphi\,(n,K(X)^n) = \max\,\{\, b(P_i),\ i=1,\dots,s\}$$

The non-compact case is much more complicated. There is a very interesting and illustrative special case: $Y=\mathbb{C}\setminus\{0,1\}$. The problem of evaluation of the amount of holomorphic functions omitting two values was first aroused by E. Gorin and V. Lin ([Z-L]) while investigating the algebraic equations on algebraic variety X. Any completely reducible polynomial $P(x,z) = \sum_0^n a_i(x)z^i$, $x\in X$, with non coinciding zeros $\lambda_i(x)$ defines n-2 functions

$\mu_1=(\lambda_1(x)-\lambda_1(x))/(\lambda_1(x)-\lambda_2(x))$ which omit values 0 and 1. E. Gorin and V. Lin offered the following

Conjecture. The number $\#H(X,\mathbb{C}\setminus\{0.1\})$ of holomorphic functions omitting two fixed values on affine variety X has an upper bound, depending on $h^1(X) = \dim H^1(X,Z)$ only.

The only two facts that are known in this direction concern one- and two-dimensional cases:

1. (V.Lin) If on affine variety X with $h^1(X) = r$ there is a function omitting r-1 values, then $\#H(X,\mathbb{C}\setminus\{0,1\}) \le (3r)^r$;

2. ([B2]) If dim X = 2, then

$$\#H(X,\mathbb{C}\setminus\{0,1\}) \le 6h^1(X)^{2(h^1(X)+h^2(X))},$$

where $h^1(X)= \dim H^1(X,Z)$.

For arbitrary variety Y, there is a theorem ([B2]) generalizing Theorem 1. Instead of ampleness of canonical bundles we introduce

Definition 1. The quasiprojective variety Y is called to be "good", if it possesses the nonsingular projectivization $\bar{Y}$, $Y=\bar{Y}\setminus D(Y)$, such that:

1) $E(Y)=K(\bar{Y})+D(Y)$ is ample;

2) D(Y) has only normal crossings.

Definition 2. We denote by A(n,d,u) the set of all n-dimensional quasiprojective varieties X with nonsingular projectivization $\bar{X}$, $X=\bar{X}\setminus D(X)$, such that

1) $E(X)=K(\bar{X})+D(X)$ is ample;

2) $E(X)^n=d$;

3) $(K(\bar{X}),E(X)^{n-1})=u$.

Theorem 2. There is such a function $\psi:Z\times Q\times Q \longrightarrow Z$, that for any "good" variety Y and for $X \in A(n,d,u)$

$$\#H(X,Y) \le \psi(n,d,u).$$

Contrary to the Theorem 1, this one does not give an answer to our main Question 1, because the numbers d and u are not known to be topological invariants of the variety X. Nevertheless, in some cases these parameters may be estimated by $h^1(X)$ (see Theorem 3 below). The proof of Theorem 2 consists of five steps.

Step I is based on the above cited Lemma of J. Kollar and T. Matsusaka. The consequence of it is that for given n, d, u there is a finite set of polynomials $\{P_1, P_2, \dots, P_s\}$, such that if $X \in A(n,d,u)$, then $\chi(X,mE(X))=P_i$ for some i, $0\le i\le s$.

Step II. Let n-dimensional variety Y be "good" , $H(X,Y) \ne \emptyset$, $f\in H(X,Y)$ and $\bar{f} : \bar{X} \longrightarrow \bar{Y}$ be it's rational extension. Consider the resolution of singularities of the mapping $\bar{f}$, i.e. such nonsingular projective variety $\tilde{X}$ and it's birational projection $\pi:\tilde{X}\longrightarrow\bar{X}$, that

1) $\pi^{-1}|X$ is isomorphic;

2) divisor $\tilde{D} = \tilde{X} \setminus \pi^{-1}(X)$ has only normal crossings as singularities;

3) in the commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\tilde{f}"] \arrow[dr, "\pi"'] & & \bar{Y} \\ & \bar{X} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

mappings $\tilde{f}$ and π are holomorphic. Since the divisors $\tilde{D}$ and D(Y) have only normal crossings,

$$\tilde{f}^{*}E(Y)=E(\tilde{X})-\tilde{R}_{f}$$

where $E(\tilde{X})=K(\tilde{X})+\tilde{D}$ and divisor R_f is effective ([I]). Hence for $m\geq 1$

$$(-1)\chi(\bar{Y},-mE(Y)) \leq \dim H^{0}(\bar{Y},(mE(Y)+K(\bar{Y})) \leq \dim H^{0}(\bar{Y},(m+1)E(Y)) \leq$$

$$H^{0}(\tilde{X},(m+1)E(\tilde{X})) \leq H^{0}(\bar{X},(m+1)E(X)) \leq (m+1)^{n}E(X)^{n}+n.$$

As $\chi(\bar{Y},mE(Y))$ is the integer valued polynomial, from this inequality follows that for given $d = E(X)^n$ there may be only finite set of polynomials $\{Q_1,..,Q_r\}$ such that $\chi(\bar{Y},mE(Y))=Q_j(m)$ for some j, $0\leq j\leq r$.

Step III. As the result of two previous steps, the finite set $\{P_1,\ldots,P_s,Q_1,\ldots,Q_r\}$ of possible characteristic polynomials is obtained, and due to the big Matsusaka Theorem ([M]), for some integer k all the divisors kE(X), kE(Y) are very ample for all the varieties $X\in A(n,d,u)$ and for "good" Y with $H(X,Y) \neq \emptyset$. Note that integer k is defined only by values of parameters n,d,u. Therefore we may assume that varieties X and Y are imbedded in the projective spaces $\mathbb{CP}^N$, $\mathbb{CP}^M$,respectively, and

$N \leq k^n d+n$

$M \leq k^n d+n$

$d(X) = \deg \bar{X} \leq k^n d$

$d(Y) = \deg \bar{Y} \leq k^n d$

$r(X) = \deg D(X) \leq k^{n-1}(d+|u|)$

$r(Y) = \deg D(Y) \leq k^{n-1}(d+|u|).$

Step IV. Considering varieties X and Y embedded as above, it is not difficult to prove, that the degree $\deg \Gamma_f$ of the graph $\Gamma_f \subset \mathbb{CP}^N \times \mathbb{CP}^M$ of any mapping $f \in H(X,Y)$ does not exceed $(n+1)k^n d = q$. To prove this we show by induction in i that

$$(E(X)^{n-i}, \bar{f}^* E(Y)^i) \le E(X)^n.$$

For i=0 it is obvious. If it is valid for $i-1 > 0$, then for i we obtain

$$(E(X)^{n-i},\bar{f}^*E(Y)^i) = (\pi^*E(X)^{n-i},\tilde{f}^*E(Y)^i) = (\pi^*E(X)^{n-i},E(\tilde{X})-R_f,\tilde{f}^*E(Y)^{i-1}) \le$$

$$(E(X)^{n-i+1},\bar{f}^*E(Y)^{i-1}) \le E(X)^n.$$

The first inequality holds because the bundles E(X) and E(Y) are ample and mappings $\tilde{f}$, $\tilde{\pi}$ - holomorphic and divisor R_f effective. Hence the intersection numbers for any $i \le n$

$$(\pi^*E(X)^{n-i}, R_f, \tilde{f}^*E(Y)^{i-1}) \ge 0.$$

Any plane section of variety $\bar{X} \times \bar{Y} \subseteq \mathbb{CP}^N \times \mathbb{CP}^M$ is equivalent to the sum $L = \sum L_1^i \times L_2^{n-i}$ where $L_1 \sim kE(X)$ and $L_2 \sim kE(Y)$. Consequently for degree $\deg\Gamma_f$ of the graph $\Gamma_f \subseteq \mathbb{CP}^N \times \mathbb{CP}^M$ of any mapping $f \in H(X,Y)$ we have

$$\deg \Gamma_f = (\Gamma_f,L) = \textstyle\sum_0^n(\Gamma_f, L_1^i \times L_2^{n-i}) = \sum (L_1,\bar{f}^*(L_2^{n-i}))$$

$$\le \textstyle\sum_0^n k^n (E(X)^i, \bar{f}^*E(Y)^{n-i}) \le (n+1)k^n E(X)^n = q$$

This estimate makes it possible to use the following Proposition ([B2], see also [Z-L]). Since the proof of this Proposition is rather technical and complicated we present it in the separate Appendix after the main text in full detail.

Proposition 1. Let $X \in A(n,d,u)$, Y is "good" and $\dim Y = n$, $X = \bar{X} \setminus D(X) \subset \mathbb{CP}^N$, $Y = \bar{Y} \setminus D(Y) \subset \mathbb{CP}^M$, and the degrees $\deg \bar{X} = d_1$, $\deg \bar{Y} = d_2$, $\deg D(X) = r_1$, $\deg D(Y) = r_2$ are fixed. Let $H^q(X,Y)$ be the set of all $f \in H(X,Y)$, for which $\deg \Gamma_f \le q$. Then the number $\#H^q(X,Y)$ of all $f \in H^q(X,Y)$ is bounded from above by constant $t(n,q,d_1,d_2,r_1,r_2,N,M)$, depending on parameters in brackets only.

Since all the parameters in our case are completely defined by n,d,u we may conclude that $\#H(X,Y) \le \psi_0(n,d,u) = t(n,q,d(X),d(Y),r(X),r(Y),N,M)$, and this finishes the proof in case $\dim X = \dim Y$.

Step V. Let now $\dim Y = n-r$. Consider the hyperplane section $\bar{S} \sim (kE(X))^r$ of $\bar{X}$, and let $S = \bar{S} \setminus D(X)|\bar{S}$. Without loss of generality we may assume that $\bar{S}$ is non-singular and $\#H(S,Y) \ge \#H(X,Y)$. Then

$$K(\bar{S}) \sim ((K(\bar{X})+rkE(X),\bar{S})$$

$$E(S) = K(\bar{S})+D(X)|\bar{S} = (1+rk)E(X)|\bar{S}$$

$$d_r = (E(S))^{n-r} = (rk+1)^{n-r}k^rE(X)^n = (rk+1)^{n-r}k^rd$$

$$u_r = (K(\bar{S}),E(S)^{n-r-1}) = (K(\bar{X})+rkE(X)|\bar{S},E(S)^{n-r-1}) =$$

$$k^r(rk+1)^{n-r-1}u+rk^{r+1}(rk+1)^{n-r-1}d.$$

Therefore

$$\#H(X,Y) \le \#H(S,Y) \le \max\,\{\psi_0(n-r,d_r,u_r),\ 0 \le r \le n\} = \psi(n,d,u)$$

for any $X \in A(n,d,u)$ and "good" Y.

As it is mentioned above, this Theorem is not the answer to the main question. The next thing to be done is to find out whether there are any restrictions imposed on values $d = E(X)^n$ and $u = (K(\bar{X}),E(X)^{n-1})$ by the topology of variety X.

The one-dimensional case is clear. For Riemann surface R of genus g with k punctures

$$d = 2g-2+k \le h^1(R),$$

$$u = 2g-2.$$

If $\dim X = 2$, there are only some partial results. Consider the nonsingular projective surface $\bar{S}$ with $K = K(\bar{S})$ and divisor D on $\bar{S}$, such that line bundle $K+D$ is ample. Let $S = \bar{S} \setminus D$.

Proposition 2. If D has only normal crossings and surface $\bar{S}$ is not ruled,

$$d = (K+D)^2 \le 4\chi(S),$$

$$u = (K,K+D) \le 4\chi(S).$$

Proof. The first inequality is well-known ([S]). On the non-ruled surface the intersection number $(K,L) \ge 0$ for any ample divisor L ([SH]). As $K+D$ is ample and D is effective, we have

$$(K,K+D) \geq 0,$$

$$(D,K+D) \geq 0,$$

$$(K,K+D) \leq (K,K+D) + (D,D+K) = (K+D)^2.$$

Q.E.D.

The requirement that D has normal crossings may be weakened, provided that $D^2 \geq 0$. We shall say that divisor D has transversal crossings, if all his singularities are non-tangent intersections and self-intersections. Any number of components is permitted to meet at one point.

Proposition 3. If $D^2 \geq 0$ and divisor D has only transversal crossings, then

$$(K+D)^2 \leq 14+120(h^1(S)+h^2(S))^2,$$

$$-40(h^1(S)+h^2(S))^2 \leq (K,K+D) \leq (K+D)^2.$$

Proof. Let $D = \sum D_i$, $D_i \neq D_j$, $G_i = \sum_{i\neq j} D_j$, $1 \leq i \leq m$, $c_i = (D_i, G_i)$. There are two possibilities: 1) Component D_i has only normal self-intersections. 2)It has some point where multiplicity of self-intersection is not less then three. In case 1) we have $h_i = h_1(D_i) \geq p_i$; if 2) $h_i \geq 2$, $(h_i)^2 + h_i \geq p_i$, where p_i denotes the arithmetic genus of the component D_i. Let s be the number of components of the first type. Then

$$r = h_1(D) \geq \sum h_i - m + \frac{1}{2} \sum c_i \geq \frac{1}{2} \sum_{i\leq s} \{(D_i, K+D_i) + c_i\} + (m-s) \geq$$

$$\frac{1}{2}\{\sum_{i\leq s} (D_i, K+D) + (m-s)\} \geq \frac{1}{2} m,$$

as $(L,K+D) > 0$ for any effective divisor L. On the other hand,

$$(D,K+D) = \sum (D_i,K+D) \le 2(\sum h_i + h_i^2) \le 4\sum h_i^2 \le 4(\sum h_i)^2 \le 4(r+m)^2 \le 40r^2.$$

From the exact sequence of the homology groups of the pair $(\bar{S},D)$ we obtain

$$h_1(\bar{S}) \le h^1(S);$$

$$r \le h^2(S) + h^1(S);$$

$$h_2(\bar{S}) \le m + h^2(S) \le 2h^1(S) + 3h^2(S)$$

Hence, arithmetic genus of surface $\bar{S}$, $|p_a| \le 1+3h^1(S)+3h^2(S)$.

From the Neuther formula: $|K^2| \le 12|p_a|+|\chi(\bar{S})| \le 14+40(h^1(S)+h^2(S))$.

So,

$$(K+D)^2=K^2-D^2+2(D,K+D) \le 14+40(h^1(S)+h^2(S))+80r^2 \le 14+120(h^1(S)+h^2(S))^2.$$

$$-(D,K+D) \le (K,K+D) \le (K+D)^2,$$

Q.E.D.

It is possible that for solving the problem in general case more delicate topological characteristics (homotopy groups, for example) are needed. There is a vast field for further investigations in this field.

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APPENDIX. The proof of Proposition 1.

Let $X = \bar{X} \setminus D(X) \subseteq \mathbb{CP}^N$, $Y = \bar{Y} \setminus D(Y) \subseteq \mathbb{CP}^M_Y$ be n-dimensional quasiprojective varieties; $\bar{X}$, $\bar{Y}$ their non-singular projectivizations, $\bar{f}\colon \bar{X} \to \bar{Y}$ – the rational extension of the mapping $f \in H(X,Y)$ and $\Gamma_f \subset \bar{X} \times \bar{Y} \subseteq \mathbb{CP}^N \times \mathbb{CP}^M$ – the graph of $\bar{f}$. Remind, that by $H^q(X,Y)$ we denote the set of such $\bar{f} \in H(X,Y)$, that $\deg \Gamma_f \leq q$.

The proof of the proposition is based on the following fundamental fact. For any projective variety $V \in \mathbb{CP}^N$ all it's n-dimensional projective subsets $W \subseteq V \subseteq \mathbb{CP}^N$ with $\deg W = q$ form an algebraic family $\{W_t\}$ and for all $t \in T$ subset W_t is defined in $\mathbb{CP}^N$ by one and the same system of algebraic equations, whose coefficients are homogeneous polynomials in variable $t \in T$. Using the fact, we obtain the following diagram of algebraic families and mappings.

$$
\begin{array}{ccccccc}
p_\Delta^{-1}(h) = D_X^h \times D_Y^h & \subset & \bar{X}_a \times \bar{Y}_a & \supset & \Gamma_f = p_\Gamma^{-1}(f) & & \\
\cap & & \cap & & \cap & & \\
\Delta & & B & & \Gamma & \subset & F \times \mathbb{CP}^N \times \mathbb{CP}^M \\
{\scriptstyle p_\Delta}\downarrow & & {\scriptstyle \pi_B}\downarrow & & {\scriptstyle p_\Gamma}\downarrow & & {\scriptstyle pr_1}\downarrow \\
H & \xrightarrow[\pi_H]{} & A & \xleftarrow[\pi_F]{} & F & \xleftarrow[pr_2]{} & F \times \mathbb{CP}^N
\end{array}
$$

In this diagram

(B,π_B,A) is the family of all the products of n-dimensional nonsingular projective varieties $\bar{X}_a \subseteq \mathbb{CP}^N$, $\bar{Y}_a \subseteq \mathbb{CP}^M$, $\deg \bar{X}_a = r_1$, $\deg \bar{Y}_a = r_2$;

(H,π_H,A) is the family of all bases $\pi^{-1}(a)$ of all the families $(\Delta_a, p_\Delta, \pi_H^{-1}(a))$

of all the products $p_\Delta^{-1}(h) = D_X^h \times D_Y^h$ of divisors $D_X^h \subset \bar{X}_a$, $D_Y^h \subset \bar{Y}_a$, $\deg D_X^h = r_1$, $\deg D_Y^h = r_2$, $\Delta_a = (p_\Delta \circ \pi_H)^{-1}(a)$;

(F, π_F, A) is the family of all the bases of the families $(\Gamma_a, p_\Gamma, \pi_F^{-1}a))$ of all the irreducible subsets $\Gamma_f \subset \bar{X}_a \times \bar{Y}_a$, $\deg\Gamma_f = q$, $\Gamma_a = (p_\Gamma \circ \pi_F)^{-1}(a)$.

pr_1, pr_2 are the natural projections.

Lemma 1. If the set $G = \{(f,h) \in F \times H$ such that

1.$\pi_H(h) = \pi_F(f) = a$;

2.$p^{-1}(f) = \Gamma_f \subset \bar{X}_a \times \bar{Y}_a$ is a graph of some mapping $f \in H^q(\bar{X}_a \setminus D_X^h, \bar{Y}_a \setminus D_Y^h)$, $D_X^h \times D_Y^h = p_\Delta^{-1}(h)\}$,

then it is algebraic.

Proof. If Γ_f defines a map $f \in H^q(X_a, Y_a)$, $X_a = \bar{X}_a \setminus D_X^h$, $Y_a = \bar{Y}_a \setminus D_Y^h$, then there is Zarisky opened subset $Z_f \subset \bar{X}_a$ of the points $z \in \bar{X}_a$, such that intersection $(\Gamma_f, \{z\} \times \mathbb{CP}^M) = 1$, i.e. the mapping f is defined in this point. At first we show that the set $\Phi = \{f \in F : \dim Z_f = n\}$ is algebraic. But the sets $\Gamma_1 = \{ \gamma \in \Gamma : \dim pr_1^{-1}(pr_1(\gamma)) = 0, \deg pr_1|pr_1^{-1}(\gamma) = 1\}$ and $\Gamma_2 = pr_1(\Gamma_1)$ are obviously algebraic; since $\Phi = \{f \in F: f \in pr_2(\Gamma_2)$ and $\dim pr_2^{-1}(f) = n\}$, it is algebraic too.

Now, $f \in \Phi$ defines the map $f \in H^q(\bar{X}_a \setminus D_X^h, \bar{Y}_a \setminus D_Y^h)$ if the following conditions hold

1. $D_X^h \times D_Y^h = p_\Delta^{-1}(h)$, $\pi_H(h) = \pi_F(a)$;

2. $\Gamma_f \cap \{D_X^h \times \mathbb{CP}^M\} \subset D_X^h \times D_Y^h$;

3. $\Gamma_f \cap \{ \mathbb{CP}^N \times D^h_Y\} \subset D^h_X \times D^h_Y$;

4. $\{z \in \overline{X}_a : \dim (\Gamma_f \cap \{z \times \mathbb{CP}^M\}) > 0 \} \subset D^h_X$.

The meaning of this inclusions is clear:

1. $D^h_X \times D^h_Y \subset \overline{X}_a \times \overline{Y}_a$;

2. $f|(\overline{X}_a \setminus D^h_X)$ is proper and surjective;

3. $f(\overline{X}_a \setminus D^h_X) \subset \overline{Y}_a \setminus D^h_Y$;

4. $f|(\overline{X}_a \setminus D^h_a)$ is holomorphic.

In order to show that the set $G \subseteq F \times H$ of all pairs (f,h) for which 1)-4) hold is quasiprojective, we are to prove that these conditions may be formulated as a system of algebraic equations and inequalities. We shall show this for 2); 3) and 4) are treated in a similar way and 1) is obvious. Let

$$\{ S^{\alpha}_f(z,w) = 0, \qquad \alpha=1,2,\dots k_1, \tag{1}$$

$$\{ V^{\beta}_h (z) = 0, \qquad \beta=1,2,\dots k_2, \tag{2}$$

$$\{ W^{\gamma}_h(w) = 0, \qquad \gamma=1,2,\dots k_3, \tag{3}$$

be the system of homogeneous in variables $z \in \mathbb{CP}^N$, $w \in \mathbb{CP}^M$, $f \in F$, $h \in H$ equations, defining the sets Γ_f, D^h_X, D^h_Y respectively. Let

$$\{\Phi^{\delta}_{h,f}(w)=0, \qquad \delta=1,2,\dots k_4 \tag{4}$$

be the system of resultants of the systems (1,2). Polynomials $\Phi^{\delta}(h,f,w)$ are homogeneous in all their variables. The set $L_Y \subset \mathbb{CP}^M$ defined by (4) consists of all the points $w \in \mathbb{CP}^M$ for which there exists $z \in \mathbb{CP}^N$ such that $(z,w) \in \Gamma_f$. The 2) is equivalent to relation

$$L_Y \subset D^h_Y,$$

that is $W^\gamma_h | L_Y = 0$ for all $\gamma=1,\dots k_4$. Denote by d^δ, Δ^γ degrees of $\Phi^\delta_{h,f}(w)$ and $W^\gamma_f(w)$ respectively. By the Hilbert theorem on zeros , there exist such numbers q^γ defined by $\{d^\delta, \Delta^\gamma\}$ and polynomials $A^{\delta,\gamma}_{h,f}$ of degrees $n^{\delta,\gamma}$ that

$$W^\gamma_h(w) = \sum_{\delta=1}^{k_4} A^{\delta,\gamma}_{h,f}(w)\, \Phi^\delta_{h,f}(w)\ , \qquad \gamma=1,\dots k_3 \tag{5}$$

and $n^{\delta,\gamma} = q^\gamma d^\delta$.

System (5) may be considered as a system of linear equations for the coefficients of polynomials $A^{\delta,\gamma}_{h,f}$. It's solvability means that the ranks of coefficients and extended matrices coincide, and this fact may be clearly expressed by the finite number of algebraic equations and inequalities for the coefficients of W^δ_h and $\Phi^\gamma_{h,f}$, which itself are homogeneous polynomials in h, f, Q.E.D.

Proof of the proposition. Any pair of varieties $X = \bar{X} \setminus D(X)$, $Y = \bar{Y} \setminus D(Y)$ considered in proposition defines points $a \in A$, $h \in H$, so that $\pi_H(h) = a$, $D^h_a = D(X)$, $D^h_a = D(Y)$, $\bar{X} = \bar{X}_a$, $\bar{Y} = \bar{Y}_a$. Since the set $H^q(X,Y)$ is finite, it is in one-to-one correspondence with the points of the set $pr^{-1}_G(h)$ where $pr_G : G \longrightarrow H$ is the natural projection of $G \subset F \times H$ to H. Since both the sets G, H are algebraic and pr_G is rational, the number $\# pr^{-1}_G(h)$ is bounded from above for all $h \in pr_G(G)$ by some constant $b(G)$ .The construction of the set G is absolutely determined by parameters n, q, d_1, d_2, r_1, r_2, N, M and we may define $t(n,q,,d_1,d_2,r_1,r_2N,M) = b(G)$, Q.E.D.

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